Electromagnetic radiation

This material is supplied for the interested student who wishes to understand the origins of some of the results contained in the course. This material is to be understood as supplementary and is designed to point the student in the right direction rather than being a complete exposition.

The level of this material is beyond the scope of what is normally included in an undergraduate chemistry degree, although it would be covered in detail in both Physics and Engineering.

Light is **electromagnetic** radiation. The wave motion of light is an oscillation of an electric and a magnetic field. These fields induce each other – according to the laws of electromagnetism a change in an electric field induces a magnetic field and vice versa. The aim of this supplementary material is to explain these statements.

Maxwell's equations

The fundamental laws of electromagnetism are expressed in terms of Maxwell's equations. There are two distinct but equivalent ways of writing these equations, a differential form and an integral form, which are related through the fundamental maths of vector calculus. We have already encountered the first equation in integral form, this is Gauss's law.

Maxwell's equations are as follows, in both integral and differential form:

1. (Gauss's law) $\oiint \vec{\mathbf{E}}.\vec{\mathbf{n}}\,dS = \frac{q}{\varepsilon_0}$ and $\nabla.\vec{\mathbf{E}} = \frac{\rho}{\varepsilon_0}$

This equation says that the electric flux through the surface enclosing a region is proportional to the charge q enclosed in the region, or that the divergence of the electric field at a point is proportional to the charge density (charge per unit volume) at that point. These forms are linked by Gauss's divergence theorem (see below).

2. (Gauss's law for magnetism) $\oiint \vec{\mathbf{B}}.\vec{\mathbf{n}}\,dS = 0$ and $\nabla.\vec{\mathbf{B}} = 0$

This equation says that the magnetic flux through the surface enclosing a region is zero, or that the divergence of the magnetic field at any point is zero. This is related to the observation that there is no magnetic equivalent to charge (magnetic monopole).

The fundamental mathematical link referred to above is Gauss's divergence theorem $\oiint \vec{\mathbf{F}}.\vec{\mathbf{n}}\,dS = \iiint \nabla.\vec{\mathbf{F}}\,dV$. The LHS is an integral over the surface A of an enclosed region, and **n** is the unit vector normal to the surface at each point. The RHS is an integral covering the whole enclosed volume. The equivalence of the two forms of Gauss's law follows by considering an infinitesimally small volume and using the divergence theorem.

3. (Faraday-Maxwell law) $\oint \vec{\mathbf{E}}.d\vec{\mathbf{s}} = -\frac{d}{dt}\iint \vec{\mathbf{B}}.\vec{\mathbf{n}}\,dA$ and $\nabla \times \vec{\mathbf{E}} = -\frac{d\vec{\mathbf{B}}}{dt}$

The LHS is a line integral around a loop and the RHS is the rate of change of the magnetic flux through any surface bounded by the loop. This equation was discovered by Faraday, who found that moving a magnet through a loop of wire caused a current to flow in the wire. The equation related the induced electric field to the rate of change of the magnetic field. At school this law becomes the rules governing dynamos. These forms are related by Stokes' theorem (see below).

4. (Ampère's law) $\oint \vec{\mathbf{B}}.d\vec{\mathbf{s}} = \mu_0\left(i_{\text{enc}} + \varepsilon_0 \frac{d}{dt}\iint \vec{\mathbf{E}}.\vec{\mathbf{n}}\,dA\right)$ and $\nabla \times \vec{\mathbf{B}} = \mu_0\left(\vec{\mathbf{j}} + \varepsilon_0 \frac{d\vec{\mathbf{E}}}{dt}\right)$

The LHS is a line integral around a loop. The RHS contains two terms. The first term is the net current flowing through a surface enclosed by the loop (and **j** is the vector current density, current per unit area). The second term is the rate of change of the electric flux through the surface. This law says that a changing electric field or a current induces a magnetic field, and implies the Biot-Savart law.

The fundamental mathematical link referred to above is Stokes' theorem $\oint \vec{\mathbf{F}}.d\vec{\mathbf{s}} = \iint \left(\nabla \times \vec{\mathbf{F}}\right).\vec{\mathbf{n}}\,dA$. The LHS is a line integral sound a loop **s.** *A* of an enclosed region, and **n** is the unit vector normal to the surface at each point. The RHS is an integral covering the whole enclosed area. The equivalence of the two forms of Faraday's or Ampère's law follows by considering an infinitesimally small area and using Stokes' theorem.

Application to Electromagnetic radiation.

This analysis can be done more economically using results from vector calculus, but these are unfamiliar to most chemists (see later).

For simplicity assume that the radiation is a plane wave propagating in the *z* direction through a vacuum containing no charge density and no current density and that the electric field has the form $\vec{\mathbf{E}}(z,t)$, representing a travelling wave in the *z* direction.

Because there is no charge density in a vacuum Gauss's law says

$$\nabla.\vec{\mathbf{E}} = \frac{\partial E_z}{\partial z} = 0$$

because the function only varies in the *z* direction. This means that the electric field does not have any component in the *z* direction, i.e. it is perpendicular to the direction of travel of the wave. Light is a *transverse* wave.

Obviously the same result holds for the magnetic field, so that both electric and magnetic fields are directed perpendicular to the direction of motion.

Now suppose that the light is plane polarised in the x direction so that $\vec{\mathbf{E}}(z,t) = E_0\phi(z,t)\vec{\mathbf{i}}$, where $\mathbf{E}_0$ is a vector in the x direction and $\phi(z,t)$ is a travelling wave moving in the z direction.

According to Faraday's law $\frac{\partial \vec{\mathbf{B}}}{\partial t} = -\nabla\times\vec{\mathbf{E}} = -E_0\begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ \phi(z,t) & 0 & 0 \end{vmatrix} = -E_0\frac{\partial\phi}{\partial z}\vec{\mathbf{j}}$. Hence if the electric field is a vector in the x direction, the magnetic field is a vector in the y direction.

According to the Ampère law $\nabla\times\vec{\mathbf{B}} = \mu_0\varepsilon_0\frac{\partial\vec{\mathbf{E}}}{\partial t}$, and differentiating with respect to time this becomes $\nabla\times\frac{\partial\vec{\mathbf{B}}}{\partial t} = \mu_0\varepsilon_0\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2}$, and for our wave,

$$\nabla\times\frac{\partial\vec{\mathbf{B}}}{\partial t} = -E_0\begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ 0 & \phi_z & 0 \end{vmatrix} = -E_0\frac{\partial^2\phi}{\partial z^2}\vec{\mathbf{i}}\text{, hence } \frac{\partial^2\phi}{\partial z^2} = -\mu_0\varepsilon_0\frac{\partial^2\phi}{\partial t^2}.$$

But this has exactly the same form as the wave equation for a travelling wave propagating with speed $c_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$.

We have therefore concluded that the fundamental equations of electromagnetism predict that in a vacuum all electromagnetic waves will propagate with the same speed, c_0.

This result can be found more generally using the rules of vector calculus, in 3d $\nabla^2\phi = -\mu_0\varepsilon_0\frac{\partial^2\phi}{\partial t^2}$.

The form of the waves

Now let us suppose that the electric field has the form $\vec{\mathbf{E}}(z,t) = E_0\cos\left(k(z-c_0t)+\phi\right)\vec{\mathbf{i}}$, in which E_0 is the amplitude of the wave, ω the angular frequency and ϕ a phase angle. We already know from Faraday's law that the variation of $\vec{\mathbf{B}}$ must be in the y direction, and it follows from Ampere's law that $\nabla\times\vec{\mathbf{B}} = \mu_0\varepsilon_0\frac{\partial\vec{\mathbf{E}}}{\partial t}$,

i.e. that

$$\nabla\times\vec{\mathbf{B}}=\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ \partial_x & \partial_y & \partial_z\\ 0 & B & 0\end{vmatrix}=-\frac{\partial B}{\partial z}\vec{\mathbf{i}}=\mu_0\varepsilon_0\frac{\partial\vec{\mathbf{E}}}{\partial t}=\mu_0\varepsilon_0E_0\frac{\partial}{\partial t}\left(\cos\left(k(z-c_0t)+\phi\right)\right)\vec{\mathbf{i}},$$

hence $\frac{\partial B}{\partial z}=-\mu_0\varepsilon_0E_0kc_0\sin\left(k(z-c_0t)+\phi\right)$

Integrating over z, $B=\mu_0\varepsilon_0E_0c_0\cos\left(k(z-c_0t)+\phi\right)=\frac{E_0}{c}\cos\left(k(z-c_0t)+\phi\right)$.

Thus the magnetic vector oscillates in phase with the electric vector, but orthogonal to it.

Intensity of light

Light carries energy in its direction of propagation, which is perpendicular to both the electric and magnetic vectors. The energy passing a point per unit time and per unit area is given by the Poynting vector,

$$\vec{\mathbf{S}}=\frac{1}{\mu_0}\vec{\mathbf{E}}\times\vec{\mathbf{B}}$$

For our plane wave this takes the form

$$\vec{\mathbf{S}}=\frac{E_0^2}{\mu_0c_0}\cos^2\left(k(z-c_0t)+\phi\right)\vec{\mathbf{k}}$$

As expected the energy travels in the z direction, and oscillates sinusoidally with the wave. The average of the cos2 term over a cycle of the wave is ½ and so we say that the average intensity of the wave is

$$I=\frac{E_0^2}{2\mu_0c_0}$$

And because $c_0^2=\frac{1}{\mu_0\varepsilon_0}$, $\frac{1}{\mu_0c_0}=\varepsilon_0c_0$, and so $I=\frac{\varepsilon_0c_0E_0^2}{2}$. This form is more convenient as it does not mix electric and magnetic quantities.

The key finding is that the intensity is proportional to the square of the amplitude of the wave.